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Non-hyperbolic surfaces having all ideal triangles with finite area

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Abstract. We construct examples of C^3 compact surfaces of non-positive curvature having non-Anosov geodesic flows and satisfying the following property: there exists $L > 0$ such that the area of every ideal triangle in the universal covering of the surface is bounded above by L.

Introduction

The subject of this paper is motivated by the following result due to J. Barge and E. Ghys [2] :

Let M be a compact surface of negative curvature. The area of ideal triangles in the universal covering $\tilde{M}$ of M is constant if and only the curvature of M is constant.

Recall that an *ideal triangle* in $\tilde{M}$ is formed by the geodesics joining 3 points $a, b, c \in \partial\tilde{M}(\infty) = S^1$, where $\tilde{M}(+\infty)$ is the usual compactification of $\tilde{M}$. Since the pull-back of any metric without conjugate points in M admits a compactification of $\tilde{M}$, it is natural to ask if the finiteness of the area of ideal triangles in $\tilde{M}$ with no conjugate points implies, for instance, that the geodesic flow of M is Anosov. The main result of this work tells us that the answer to this question is negative, even in the category of surfaces of non-positive curvature:

Theorem 1. *Given $\beta > 1$ there exist $\alpha > 0$, $C, D > 0$ and a $C^{2+\alpha}$ compact surface M having negative curvature at all points but along a simple closed geodesic $\gamma(t)$-where the curvature is zero at every point-*

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such that:

1. *The geodesic flow of M is expansive.*
2. *The stable geodesics $\gamma_s(t)$ of $\gamma(t)$ satisfy*

$$d(\gamma(t), \gamma_s(t)) \leq \frac{C}{t^\beta} d(\gamma(0), \gamma_s(0))$$

   for every $t \geq 0$.
3. *The area of every ideal triangle in $\tilde{M}$ is bounded above by D.*

The equation relating α and β in Theorem 1 is $\alpha = \frac{2}{\beta}$, so if $\beta = \frac{3}{2}$ the corresponding surface is of class C^3, and thus we obtain the desired counter example. Notice that item 1 of Theorem 1 is a direct consequence of the fact that the surface does not have flat strips. So, in particular, the geodesic flow of such surfaces is equivalent to an Anosov flow (notice that this is a necessary condition for the existence of these surfaces since the presence of flat strips obviously implies that the area of ideal triangles is not bounded).

Theorem 1 gives us curious examples of objects of interest in the areas of dynamical systems and Riemannian geometry. For, on the one hand, the geodesic flow of the C^3 example of Theorem 1 is a C^2 expansive, non-Anosov flow having almost every Lyapunov exponent different from zero (and thus the asymptotic behavior of almost every tangent vector is exponential). However, the asymptotic behavior of orbits in the center stable submanifold of the geodesic $\gamma(t)$ of the statement of Theorem 1 is of the order of $\frac{1}{t^2}$, still granting the finiteness of the area of ideal triangles having liftings of $\gamma(t)$ as edges which is of the order of $\int_0^{+\infty} \frac{1}{t^2} dt$. A priori, the non-vanishing of Lyapunov exponents of almost every tangent vector does not ensure the finiteness of the area of positively saturated pieces of center stable submanifolds. Recall that in the Anosov case, the area of ideal triangles is of the order of $\int_0^{+\infty} e^{-at} dt$ for some $a > 0$.

On the other hand, there are no C^∞ examples of such surfaces. Namely, a non-positively curved C^4 surface satisfying the hypotheses in the statement of Theorem 1 has the property that the area of every ideal triangle having a lifting of $\gamma(t)$ as an edge has infinite area, as proved in [**10**]. So although the answer to our initial question is negative, to get a counter example we had to break down the differentiability of the surface, and C^3 is the best possible class of differentiability of these family of examples.

The paper has four sections. In section 1 we construct a family of examples of surfaces of non-positive curvature having a closed, non-hyperbolic geodesic with certain special properties. The idea is to deform an annulus of revolution of curvature -1 near the systole of the annulus to obtain a another systole with some prescribed non-hyperbolic, expansive, asymptotic behaviour of the stable manifold. To determine this new metric in the annulus, we use the well-known differential equations for the geodesics in surfaces of revolution. The key step at this stage is that we can solve the differential equation for the geodesics by prescribing the rate at which asymptotic geodesics approach the systole. This is done without any previous data on the Riemannian metric on the annulus. Then, from the solutions of the equations we shall construct a metric whose geodesics have the required non-hyperbolic dynamics, and finally we glue this new annulus with a surface of curvature -1. The remainder of the paper is devoted to the proof of the finiteness of the area of ideal triangles of such surfaces. In section 2 we show that the problem reduces to analyze the area of thin ends of ideal triangles. In section 3 we estimate the area of the intersections of these ends with a tubular neighborhood containing the zero curvature geodesic, and finally in section 4 we estimate the area of the pieces of the ends in a "good" region of negative curvature. Putting together the estimates of sections 3 and 4 we conclude the proof of Theorem 1.

1. The construction of the metrics

Let us fix some notations. If M is a Riemannian manifold, $\tilde{M}$ will denote its universal covering endowed with the pull-back of the metric of M. The Gaussian curvature at a point $p \in M$ will be $K(p)$. All geodesics will be parameterized by arc length throughout the paper. The main result of the section is the following:

Proposition 1.1. *Given $\beta > 1$ there exists an annulus A of revolution in*

R^3 with a $C^{2+\frac{2}{\beta}}$ Riemannian metric satisfying the following conditions:

1. *The curvature of the annulus is negative in all points but along a simple, closed geodesic $\gamma_0(t)$ where the curvature is zero $\forall t$.*
2. *The annulus A is symmetric with respect to $\gamma_0(t)$.*
3. *On every point $p \in A$ there exists a unique geodesic $\eta_p(t)$ asymptotic to $\gamma_0(t)$ such that*
$$d(\gamma_0(t), \eta_p(t+b)) \leq \frac{D}{t^\beta}$$
for some b and D independent of p.
4. *There exists Q depending on β such that if $\bar{\gamma}(t)$ and $\bar{\eta}_p(t)$ are two liftings of γ_0 and η_p with*
    - $\lim_{t\to\infty} d(\bar{\gamma}(t), \bar{\eta}_p(t)) = 0.$
    - $d(\bar{\gamma}(0), \bar{\eta}_p(0)) \leq 1$ *then the area of the region in $\tilde{M}$ bounded by $\bar{\gamma}([0,+\infty))$, $\bar{\eta}_p([0,+\infty))$ and the geodesic segment joining $\bar{\gamma}(0)$ and $\bar{\eta}_p(0)$ is less or equal than Q.*

Remark. We actually give the rate at which the curvature approaches to zero near the geodesic $\gamma(t)$. Indeed, we show that

$$r(h) = 1 + C(\beta)h^{2+\frac{2}{\beta}} + o(h^{2+\frac{2}{\beta}}),$$

where $r(h)$ is the generating function of the annulus A of revolution, and $f(x) = o(x)$ means that $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

Proof. Let $r(h)$ be the generating function of an annulus of revolution. Recall that $r(h)$ gives the radii of the parallels of A. Let us chose the z axis to be the axis of revolution. In this way, if $u_\theta = (\cos(\theta), \sin(\theta))$ then the surface is given in coordinates by

$$f(r,\theta) = (ru_\theta, h(r)).$$

We are interested in determining $r(h)$ in a neighborhood $\mid h \mid < \epsilon$ satisfying the following conditions:

1. $r(h)$ is at least of class C^2.
2. $r(0) = 1$ is a strict minimum of r and $r(h)$ is convex in $\mid h \mid < \epsilon$.

With these conditions it is easy to see that the parallel $\gamma_0(\theta) = (r(0)u_\theta, 0)$ is the only parallel which is a geodesic in $\mid h \mid < \epsilon$. Moreover, the curvature of A is negative at every point close to γ_0 and through every point in this neighborhood exists an asymptote of γ_0, i.e., a geodesic $\eta(t)$ satisfying

$$\lim_{t\to+\infty} d(\gamma_0(t),\eta(t)) = 0.$$

To start with the construction of $r(h)$ we shall impose the fact that γ_0 is a geodesic having asymptotes (notice that parallels are not geodesics in general) in the differential equations of the geodesics of surfaces of revolution. So from now on $\eta(t)$ will be asymptotic to γ_0 and $\eta(t) = (r(t)u_{\theta(t)}, h(t))$ will be its expression in coordinates, where $r(t) = r(h(t))$. We briefly recall how to deduce the equations of geodesics in surfaces of revolution. We have that

$$\eta'(t) = (r'u_\theta + r\theta' u_\theta^\perp, h')$$

$$\eta''(t) = ((r'' - r(\theta')^2)u_\theta + (2r'\theta' + r\theta'')u_\theta^\perp, h''),$$

where $u_\theta^\perp = u_{\theta+\frac{\pi}{2}}$ and $f' = \frac{df}{dt}$. If $h'(t) \neq 0$ then the vector $(\frac{dr}{dh}u_\theta, 1)$ tangent to the annulus is parallel to $(r'u_\theta, h')$, hence the tangent space at $\eta(t)$ is

$$T_{\eta(t)}A = span\{(u_\theta^\perp, 0), (r'u_\theta, h')\}.$$

The fact that $\eta(t)$ is a geodesic is equivalent to $\eta''(t)$ is perpendicular to $T_{\eta(t)}A$. Therefore we get

$$2r'\theta' + r\theta'' = 0 \tag{1}$$

and

$$r'r'' - rr'(\theta')^2 + h'h'' = 0. \tag{2}$$

The first of them implies that $\theta' = \frac{k}{r^2}$ for some constant k (this is a version of the well-known Clairaut equation). We want that $k = 1$ for γ_0, and since we want η to be asymptotic to γ_0 then $k = 1$ for η also. Notice that this also implies that $\gamma_0 = \{h = 0\}$ is a geodesic. Putting together these two equations we obtain

$$\frac{d}{dt} \mid \eta'(t) \mid^2 = \frac{d}{dt}((r')^2 + r^2(\theta')^2 + (h')^2) = 0$$

i.e., (as we should expect) $\mid \eta'(t) \mid$ is constant. We take this constant equal to 1, and replacing $(\theta')^2 = \frac{1}{r^2}$ in this equation we obtain

$$(r')^2 + \frac{1}{r^2} + (h')^2 = 1. \tag{3}$$

Let us look at the blowing up of this equation at $+\infty$. Let $s = \frac{1}{t}$, so

$$\frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = -s^2 r_s,$$

where $r_s = \frac{dr}{ds}$. Then the above equation becomes

$$s^4 r_s^2 + \frac{1}{r^2} + s^4 h_s^2 = 1$$

and hence we get

$$h_s^2 = \frac{r^2 - 1}{r^2 s^4} - r_s^2.$$

The key step of the proof is the following:

Claim: Assume that $r(s) = 1 + s^\alpha$ for some $\alpha \geq 4$. Then there exists h of the form

$$h(s) = A(\beta)s^\beta + o(s^\beta),$$

where $\beta = \frac{\alpha-2}{2}$, satisfying the equation $h_s^2 = \frac{r^2-1}{r^2 s^4} - r_s^2$.

Indeed, replacing $r(s)$ in the equation we get

$$\begin{aligned} h_s^2 &= \frac{2s^\alpha + s^{2\alpha}}{(1+s^\alpha)^2 s^4} - \alpha^2 s^{2\alpha-2} \\ &= \frac{2s^{\alpha-4} + s^{2\alpha-4}}{(1+s^\alpha)^2} - \alpha^2 s^{2\alpha-2}, \end{aligned}$$

which is positive and finite if $\alpha \geq 4$ and s small enough, and this is enough to grant the existence of h_s. We can write the above equation as

$$h_s^2 = \left(\frac{s^{\frac{\alpha}{2}-2}}{1+s^\alpha}\right)^2 (2 + s^\alpha - \alpha^2 s^{\alpha+2}(1+s^\alpha)^2)$$

so we obtain

$$h_s = \left(\frac{s^{\frac{\alpha}{2}-2}}{1+s^\alpha}\right) (2 + s^\alpha - \alpha^2 s^{\alpha+2}(1+s^\alpha)^2)^{\frac{1}{2}}$$

Since we want $\eta(t)$ to be asymptotic to the parallel $h = 0$ we use the condition $h(s = 0) = 0$ and we get

$$h(s) = \int_0^s \left(\frac{x^{\frac{\alpha}{2}-2}}{2 + x^\alpha}\right)(1 + x^\alpha - \alpha^2 x^{\alpha+2}(1 + x^\alpha)^2)^{\frac{1}{2}}\,dx.$$

This integral is finite for every $\alpha \geq 4$, and from it we can deduce that

$$h(s) = A(\beta)s^\beta + o(s^\beta),$$

where $\beta = \frac{\alpha}{2} - 2 + 1 = \frac{\alpha-2}{2}$ and $\lim_{s\to 0} \frac{o(s^\beta)}{s^\beta} = 0$. This concludes the proof of the claim.

Going back to t, we replace $s = \frac{1}{t}$ and get

$$\begin{aligned} r(t) &= 1 + \frac{1}{t^\alpha} \\ h(t) &= A(\beta)\frac{1}{t^\beta} + o(\frac{1}{t^\beta}) \\ \theta(t) &= \theta(T_0) + \int_{T_0}^t \frac{1}{r^2(\tau)}d\tau \end{aligned} \tag{4}$$

for $t \geq T_0$ and T_0 suitably large. By construction, the above functions satisfy the differential equations for the geodesics of the surface. If s is small enough we have that $\frac{dh}{ds} > 0$ hence $\frac{dh}{dt} < 0$ for t large. In particular, the map $t \mapsto h(t)$ is a diffeomorphism from $[T_0, +\infty)$ to $[0, h(T_0))$ for T_0 large. We can invert it to get $r(h) = r(t(h)) = 1 + s(h)^\alpha$, with $r(h = 0) = 1$. Extend r to $h < 0$ by $r(h) = r(-h)$. For $h > 0$ we have that

$$s = C(\beta)h^{\frac{1}{\beta}} + o(h^{\frac{1}{\beta}})$$

and

$$\begin{aligned} r(h) &= 1 + D(\beta)h^{\frac{\alpha}{\beta}} + o(h^{\frac{\alpha}{\beta}}) \\ &= 1 + D(\beta)h^{2+\frac{2}{\beta}} + o(h^{2+\frac{2}{\beta}}). \end{aligned}$$

The function $r(h)$ defines an annulus of revolution which is $C^{2+\frac{2}{\beta}}$ if $\beta > 1$. Observe that $\frac{d^2r}{dh^2} > 0$ if $h \neq 0$ and $\frac{d^2r}{dh^2}(0) = 0$, so $r(h)$ is in fact a convex function with an isolated minimum at $h = 0$. Since $\theta' = 1$, $r(t) = 1$, $h(t) = 0$ satisfy equations (2) and (3), we have that

$\gamma_0 = \{h = 0\}$ is a geodesic. The solution given in (4) determines a geodesic $\eta(t) = (r(t)u_{\theta(t)}, h(t))$ which is asymptotic to γ_0. Actually, by rotational symmetry of the annulus we see that through every point of the annulus there exists a (unique) geodesic which is asymptotic to γ_0, and this geodesic is isometric to $\eta(t)$. The symmetry with respect to $h = 0$ is a consequence of the construction of the function $r(h)$.

We now proceed to estimate the rate at which the asymptotes of γ_0 approach γ_0. Let $r(t)$, $h(t)$ and $\theta(t)$ be as before, and let $\gamma_b(t) = (u_{t+b}, 0)$, where b is a constant to be determined later. Let $a > 0$ be such that $| r | < 2$, $| r_h | < 2$ for every $| h | < a$. Then we have

$$\begin{aligned} d(\eta(t), (u_{\theta(t)}, 0)) &\leq \int_0^{h(t)} (1 + r_h^2)^{\frac{1}{2}} dh \\ &\leq \int_0^{h(t)} (1 + 4)^{\frac{1}{2}} dh \\ &\leq 3h(t) \\ &\leq 3(1 + A(\beta)) \frac{1}{t^\beta} \end{aligned}$$

and

$$d(\gamma_b(t), (u_{\theta(t)}, 0)) \leq t + b - \int_{T_0}^t \frac{1}{r^2(t)} dt$$

We will show that b can be chosen such that

$$\left| t + b - \int_{T_0}^t \frac{1}{r^2(t)} dt \right| \leq E \frac{1}{t^\beta}$$

for some constant $E > 0$. Since

$$d(\eta(t), \gamma_b(t)) \leq d(\eta(t), (u_{\theta(t)}, 0)) + d((u_{\theta(t)}, 0), \gamma_b(t))$$

this will provide the estimate required in the statement of the proposition. Indeed, there is a constant $H(T_0)$ such that

$$
\begin{aligned}
\left| t + b - \int_{T_0}^{t} \left(\frac{1}{1+\frac{1}{x^\alpha}} \right)^2 dx \right| &= \left| b + T_0 + \int_{T_0}^{t} (1 - \frac{x^{2\alpha}}{(1+x^\alpha)^2}) dx \right| \\
&\leq \left| b + T_0 + \int_{T_0}^{t} \frac{2}{x^\alpha} + \frac{1}{x^{2\alpha}} dx \right| \\
&\leq \left| b + T_0 + \frac{-4}{(\alpha-1)t^{\alpha-1}} \Big|_{T_0}^{t} \right| \\
&= \left| b + H(T_0) + \frac{-4}{(\alpha-1)t^{\alpha-1}} \right|
\end{aligned}
$$

so taking $b = -H(T_0)$ we obtain the estimate.

Finally, we estimate the area between any two asymptotic liftings $\bar{\gamma}_0(b+t)$ and $\bar{\eta}(t)$ in $\bar{A}$ of $\eta(t)$ and $\gamma_0(b+t)$. Consider the parameterization $F : R \times [-a, a] \longrightarrow A$ given by $F(\theta, h) = (r(h)u_\theta, h)$. Then

$$
\frac{\partial F}{\partial \theta} = (r u_\theta^\perp, 0)
$$

$$
\frac{\partial F}{\partial h} = (r_h u_\theta, 1)
$$

and its Jacobian is

$$
Jac(F) = r + r r_h,
$$

which is less than 6 if $\mid h \mid < a$ where a was chosen before. If $\beta > 1$ the area of the region bounded by $\bar{\gamma}_0$, $\bar{\eta}$ for and the geodesic joining $F(\theta(T), h(T))$ with $F(\theta(T), 0)$ is bounded above by

$$
\begin{aligned}
\int_{\theta(T)}^{+\infty} \int_0^{h(T)} Jac(F) dh d\theta &\leq \int_{\theta(T)}^{+\infty} \int_0^{h(T)} 6 dh d\theta \\
&\leq \int_{\theta(T)}^{+\infty} 6h(t)\theta'(t) dt \\
&= \int_{\theta(T)}^{+\infty} \frac{6}{r^2(t)} h(t) dt \\
&\leq \int_{\theta(T)}^{+\infty} 6h(t) dt \\
&= 6 \int_{\theta(T)}^{+\infty} \frac{A(\beta)}{t^\beta} + o(\frac{1}{t^\beta}) dt
\end{aligned}
$$

which is finite for $\beta > 1$. The difference between this region and the region bounded by $\bar{\gamma}_0(t+b)$, $\bar{\eta}(t)$ for $t \geq T$ and the stable-horospheric segment joining $\bar{\gamma}_0(T)$ with $\bar{\eta}(T)$ is a compact region whose area is bounded by some constant (not depending on η) times the distance $d(\gamma_0(T), \eta(T)) \leq E\frac{1}{T^\beta}$ which completes the proof of the proposition. □

Corollary 1.1. *Given $\beta > 1$ there exists a compact surface of negative curvature at all points but at the points of a simple closed geodesic γ_0, where the curvature is zero, satisfying the conclusions of Proposition 1.1 in a tubular neighborhood N of γ_0.*

Proof. The argument is quite standard, so we just give an idea of the proof. Given $L > 0$ there exists an annulus of revolution of negative curvature -1 in R^3 having a simple closed geodesic of length L which minimizes the length of closed path in the annulus. This annulus is symmetric with respect to this geodesic. Let $R = R(L)$ be the radius of the closed geodesic, let us assume that the axis of the annulus is the z-axis and that this geodesic is in the horizontal plane. So the annulus in coordinates will be $(s(z)u_\theta, z)$ for some convex function $s(z)$ and the closed geodesic will be given by $z = 0$. Recall that $r(z)$ is the function we found in Proposition 1.1 generating an annulus of revolution with some special properties. To find $r(z)$ we gave $r(0) = 1$ as an initial condition for the problem, but in fact it is possible to find such a function with $r(0) = r_0 > 0$. If $R < r_0$ (i.e., $L < 2\pi r_0$) it is possible to deform the generating curve $s(z)$ to a convex curve $\bar{s}(z)$ which coincides with $r(z)$ in a neighborhood of $z = 0$. Now, if S is a compact surface of constant negative curvature -1 it is possible to embed isometrically in R^3 an annulus $\bar{A}$ contained in S symmetric with respect to some closed geodesic. Hence, we "cut" $\bar{A}$ from S, we deform it in R^3 to obtain a copy of the annulus A of Proposition 1.1 glued with a piece of $\bar{A}$, and then we glue this new annulus with $S - \bar{A}$ to obtain the required surface. □

2. Reduction to thin ends of ideal triangles

Let $M = (M, g)$ be a compact surface and let $\tilde{M}$ be its universal covering endowed with the pullback of the metric g by the covering map. By an *end* in $\tilde{M}$ we shall mean a subset of $\tilde{M}$ bounded by two asymptotic geodesics and a segment of stable horosphere connecting these geodesics. The *width* of an end will be the Hausdorff distance between the geodesics in its boundary. The purpose of this section is to show the following result:

Proposition 2.1. *Let M be a compact surface of non-positive curvature having no flat strips. Then for a given $\rho > 0$ there exists $r > 0$ such that every ideal triangle Δ in $\tilde{M}$ satisfies the following:*

There exists a ball B in $\tilde{M}$ of radius r such that the complement of $\Delta \cap B$ consists of three disjoint strips bounded by asymptotic geodesics having widths less than ρ.

To begin with the proof let us first fix some notations. A complete metric space (X, d) is said to be *geodesic* if every two points p, q in the space are joined by an isometric immersion of the interval $[0, d(p, q)]$ endowed with the Euclidean metric. In general, we shall refer to a geodesic segment as $[p, q]$. The geodesics in the surface will be always parameterized by arclength. Now, let us recall the notion of a Gromov-hyperbolic space [**7**]:

Definition 1. Let (X,d) be a complete, geodesic metric space. A geodesic triangle with vertices a_0, a_1, a_2 is said to be δ-thin if for every $p \in [a_i, a_{i+1}]$ we have that

$$d(p, [a_{i+1}, a_{i+2}] \cup [a_{i+2}, a_i]) \leq \delta,$$

where the indices above are taken mod 3. The space (X, d) is called δ-hyperbolic or Gromov-hyperbolic if every geodesic triangle is δ-thin for some δ.

Lemma 2.1. *Let M be a compact surface of genus greater than two with no conjugate points. Then $\tilde{M}$ is δ-hyperbolic for some δ depending on the metric.*

Proof. The proof is indeed well-known and relies in the following two facts:

1. The Poincaré plane is a δ_0 hyperbolic space where δ_0 depends on the Poincaré metric.

2. Let M be a compact surface of genus greater than two having no conjugate points. Then there exists D > 0 such that every geodesic of $\tilde{\mathrm{M}}$ is contained in a tubular neighborhood of a geodesic of the Poincaré plane. This fact is essentially due to Morse [**8**] but there are more general and recent versions by Eberlein [**5**] for visibility manifolds and Gromov for δ-hyperbolic spaces. Now, it is not hard to see that $\tilde{M}$ is δ hyperbolic for some δ depending on δ_0 and D.

Corollary 2.1. *Let M be a compact surface of non-positive curvature and genus greater than 1. Given any ideal triangle with vertices at infinity a_0, a_1, a_2 there exist points $p_i \in [a_i, a_{i+1}]$ such that*

$$d(p_i, p_j) \leq 2\delta$$

for every i and j mod 3.

Proof. The arguments of this proof are quite standard in the theory of δ-hyperbolic spaces (see [**7**] for instance). Nevertheless we shall sketch the proof for the sake of completeness. Since every ideal triangle in $\tilde{\mathrm{M}}$ can be obtained as a limit of a sequence of geodesic triangles with vertices in $\tilde{\mathrm{M}}$ we get that for every p in $[a_i, a_{i+1}]$ the inequalities of definition 2.1 hold:

$$d(p, [a_{i+1}, a_{i+2}] \cup [a_{i+2}, a_i]) \leq \delta.$$

So consider for instance the geodesic $\gamma = [a_0, a_1]$ and let $\bar{p} \in [a_0, a_1]$ be such that $d(\bar{p}, [a_0, a_2]) \leq \delta$. Parametrize γ by arclength in a way that $\gamma(0) = \bar{p}$ and $\lim_{t\to+\infty} \gamma(t) = a_2$. Let us denote $[a_0, a_2] = \beta$ and let us parametrize β in a way that $\beta(t)$ is the intersection of the stable horosphere of $\gamma(t)$ with β. Since the distance $d(t) = d(\gamma(t), \beta(t))$ is an increasing function there exists $p_0 = \gamma(t_0)$ such that

$$t_0 = \sup\{t \in R, d(\gamma(t), \beta(t)) \leq \delta\ \forall t \leq t_0\}.$$

So we have that $d(p_0, q) \geq \delta$ for every $q \in \beta = [a_0, a_2]$ and that there exists a point $p_2 \in [a_0, a_2]$ such that

$$d(p_0, p_2) \leq \delta.$$

On the other hand, by the definition of δ hyperbolicity we deduce that for every $\gamma(t)$ with $t > t_0$ there must exist $p(t) \in [a_1, a_2]$ such that

$$d(\gamma(t), p(t)) \leq \delta.$$

By approaching $p_0 = \gamma(t_0)$ with a sequence $\gamma(t_n), t_n > t_0$ we get a point $p_1 \in [a_1, a_2]$ such that $d(p_0, p_1) \leq \delta$. This implies that

$$d(p_1, p_2) \leq d(p_1, p_0) + d(p_0, p_2) \leq 2\delta,$$

which finishes the proof of the statement. □

Next, we show a particular property of expansive geodesic flows in manifolds with no conjugate points (see [**9**] for instance):

Lemma 2.2. *Let M be a compact manifold of non-positive curvature and no flat strips. Then the distance between asymptotic geodesics contracts uniformly in $\tilde{M}$, i.e., given $\epsilon > 0$, $0 < \eta < \epsilon$ there exists $R > 0$ such that if $\gamma(t), \beta(t)$ are geodesics in $\tilde{M}$ satisfying:*

$$d(\gamma(t), \beta(t)) \leq D \ \forall t \geq 0$$

then there exists parameterizations of γ and β such that
1. $d(\gamma(0), \beta(0)) = \epsilon$.
2. $d(\gamma(t), \beta(t)) \leq \eta, \quad \forall t \geq R$.

Proof. To simplify things we adopt the following convention: let $\gamma(t)$ be a geodesic in $\tilde{\mathrm{M}}$, then every asymptote $\beta(t)$ of $\gamma(t)$ is parameterized in a way that $\beta(t)$ is the point of intersection between β and the stable horosphere of $\gamma(t)$. Now, observe that if $\gamma(t)$ and $\beta(t)$ remain at bounded distance D for every positive t we can define $\gamma_n(t), \beta_n(t)$ for $n \geq 1$ by

$$\gamma_n(t) = \gamma(n+t), \ \beta_n(t) = \beta(n+t),$$

so we have that

$$d(\gamma_n(t), \beta_n(t)) \leq D$$

for every $-n \leq t$. Thus, up to isometries in $\tilde{\mathrm{M}}$ we have a convergent subsequence of the $\gamma_n \longrightarrow \gamma_0(t)$, $\beta_n \longrightarrow \beta_0(t)$ such that
1. $d(\gamma_0(t), \beta_0(t)) \leq D$ for every t.
2. $d(\gamma_0(0), \beta_0(0)) = \lim_{n\to+\infty} d(\gamma_n(0), \beta_n(0)) = \lim_{n\to+\infty} d(\gamma(n), \beta(0))$

Since the curvature is non-positive, the distance function between two geodesics is a convex function, therefore it has to be constant in this case. And if γ_0 and β_0 are different they have to bound a flat strip, contradicting the assumptions. This implies that

$$\lim_{n\to+\infty} d(\gamma(n), \beta(n)) = 0$$

showing that the distance between two asymptotic geodesics is a contraction. Moreover, since the distance is decreasing it must assume the value ϵ somewhere. Let us suppose that $d(\gamma(0), \beta(0)) = \epsilon$. So, if item 2 in the statement was not true it would exist a sequence of asymptotic geodesics $\alpha_n(t), \theta_n(t)$ satisfying:

1. $d(\alpha_n(0), \beta_n(0)) = \epsilon$.
2. $d(\alpha_n(t), \theta_n(t)) \geq \eta\ \forall 0 \leq t \leq n$ and by exactly the same previous argument we would obtain a pair of geodesics $\alpha_0(t), \theta_0(t)$ with

$$d(\alpha_0(t), \theta_0(t)) \leq \epsilon$$

and

$$d(\alpha_0(0), \theta_0(0)) \geq \eta$$

contradicting again the absence of flat strips in M. □

Proof of Proposition 2.1

So let a_0, a_1, a_2 any three points in the ideal boundary of $\tilde{M}$. Let $\gamma_0 = [a_0, a_1]$, $\gamma_1 = [a_1, a_2]$ and $\gamma_2 = [a_2, a_0]$. Let p_0, p_1, p_2 as in corollary 2.1. As a consequence of the previous results we have that given any $\rho < 2\delta$ there exists $R > 0$ such that if $q \in \gamma_i$ is at distance greater than R from p_i then either

$$d(q, \gamma_{i+1}) \leq \rho$$

or

$$d(q, \gamma_{i+2}) \leq \rho$$

where the indices are taken mod 3. Taking (without loss of generality) $R \geq 4\delta$ and letting $r = 2R$ it is easy to check by the triangle inequality that the complement of the ideal triangle whose vertices are a_0, a_1, a_2 with respect to the ball of radius r centered at p_0 satisfies the assertion in the statement of Proposition 2.1. □

3. The behaviour of thin ends in the critical region

In this section we are going to consider our examples of surfaces of non-positive curvature constructed in the first section. From now on the surface M will be such a surface. The goal of the section is to estimate the area of the intersections of the ends of ideal triangles with the critical region of the surface, i.e., the region near the liftings in $\tilde{M}$ of the zero curvature closed geodesic. We shall fix a lifting of this geodesic and let us denote it again by γ_0. Let us start by choosing a special tubular neighborhood of γ_0 in $\tilde{M}$. Recall that M is constructed by gluing an annulus of revolution with a negatively curved surface with boundary.

Lemma 3.1. *There exists $\omega > 0$ smaller than the injectivity radius of M such that in the tubular neighborhood N of radius $\frac{\omega}{2}$ of $\gamma_0(t)$ we have*

1. *There exists a one parameter family of rotations preserving N acting on γ_0 by translations. The rotations are isometries of N.*
2. *There exists a reflection fixing γ_0 preserving N which is an isometry of N.*
3. *Every geodesic staying in N during an infinite interval of time is an (stable or unstable) asymptote of γ_0.*

Proof. Items 1 and 2 in the statement follow from the construction in section 1: a surface of revolution is invariant by a one parameter family of rotations preserving the parallels of the surface. Item 3 follows from the fact that the geodesic flow of the surface is expansive, so there exists an expansiveness constant satisfying this statement. Thus, any ω smaller than both the injectivity radius of the annulus of the construction and the expansiveness constant of the geodesic flow satisfies the conditions in the statement. □

Remark that the rotations of the annulus preserve the stable (unstable) character of geodesics with respect to γ_0. So stable (unstable) geodesics of γ_0 in N are obtained by the action of the rotations over any single stable (unstable) geodesic. In particular, the angle of first intersection ϵ_0 of unstable geodesics of γ_0 with the boundary ∂N of N is constant, independent of the geodesic. Similarly, the angle of last intersection of stable geodesics with ∂N is constant along this boundary and from the symmetries of the annulus it is also equal to ϵ_0. From the uniqueness of geodesics in terms of their initial conditions, a geodesic is an asymptote of γ_0 if and only if it intersects eventually ∂N with an angle ϵ_0. Next, we state a technical lemma concerning the description of geodesics crossing the neighborhood N with an angle different from ϵ_0. Let N_1, N_2 be the connected components of ∂N.

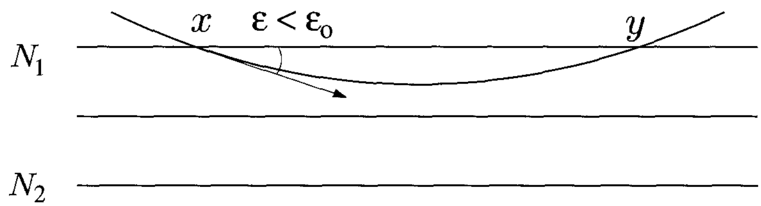

Figure 1

Lemma 3.2. *Given $0 < \epsilon < \epsilon_0$ there exists $k = k(\epsilon) > 0$ such that every geodesic segment [x,y] satisfying:*

1. *The points x and y belong to ∂N.*
2. *The segment [x,y] is a subset of N.*
3. *The angle between [x,y] and ∂N at x is less than ϵ has length less than k. Moreover, x and y belong to the same connected component of ∂N, [x,y] remains in a region of negative curvature and $\lim_{\epsilon \to 0} k(\epsilon) = 0$.*

Proof. It is straightforward from the above remarks. First of all it is clear that every geodesic intersecting ∂N with an angle different from ϵ_0 must leave in finite time the region N, otherwise it would be an asymptote of γ_0 contradicting the choice of ϵ. Moreover, this exit time does not depend on the geodesic but on the angles of intersection with ∂N. Otherwise, we would obtain a sequence of geodesic segments of increasing lengths all contained in N and intersecting ∂N with an angle less than ϵ_0, from which we get a convergent subsequence whose limit would be an asymptote of γ_0 having a "wrong" intersection angle with ∂N.

To show that x and y must belong to the same connected component of ∂N just remark that if the angle of intersection at x is less than ϵ_0 then $[x, y]$ is locally closer to ∂N than the two asymptotes of γ_0 through x crossing ∂N with angle ϵ_0.

Claim:. The segment $[x, y]$ cannot cross γ_0.

Otherwise it would cross one of the asymptotes through x at least twice in the region N because stable geodesics do not meet γ_0 after intersecting ∂N with angle ϵ_0 (analogously for unstable geodesics before first intersection with ∂N). But this is not allowed by the choice of N and the fact that M has nonpositive curvature.

Thus, since y belongs to ∂N it has to be in the same component of x. Of course this implies that $[x, y]$ remains in a region of negative curvature by the construction of M. □

Now we are ready to estimate the area of crossings of thin ends of triangles with the critical region. Define

- A positive number α_0 given by the following property: let $\gamma(t), \beta(t)$ be two *asymptotic* geodesics in M such that $d(\gamma(0), \beta(0)) \leq \alpha_0$. Then, if $\gamma(t)$ crosses ∂N at $\gamma(t_0)$ making an angle greater than $\frac{\epsilon_0}{2}$ then $\beta(t)$ also crosses ∂N at some $\beta(s_0)$ near $\gamma(t_0)$.
- Let $\alpha_1 = \min\{\alpha_0, \frac{\omega}{4}\}$, where ω is the constant of lemma 3.1.

There exists r depending on α_1 such that the complement of any ideal triangle with respect to some ball of radius r consists of three disjoints strips of widths less than α_1. The area of the region of any ideal triangle inside these balls is bounded above by the area of the balls which depends only on r. So we are left with the estimates of areas of ends whose widths are less than α_1. Again, we shall assume than given one geodesic $\gamma(t)$ all the asymptotes β of γ contained in a tubular neighborhood of radius α_1 are parameterized by the stable horospheres of $\gamma(t)$, i.e., if $\beta(t)$ is the intersection of the stable horosphere of $\gamma(t)$ with β.

So let E be one of these ends bounded by two geodesics γ_1 and γ_2. To avoid complications in the notation, we shall identify E with its projection in M by the covering map, and we shall suppose that E has a piece of stable horosphere of γ_i in its boundary. In other words, let us assume that ∂E has three parts:

1. $\gamma_1(t)$ for $t \geq 0$.
2. $\gamma_2(t)$ for $t \geq 0$.
3. The segment of stable horosphere of $\gamma_i(0)$ for i=1, 2 with endpoints $\gamma_1(0)$ and $\gamma_2(0)$.

The main result of this section is the following:

Proposition 3.1. *Let E be an end of an ideal triangle having the properties stated above. Let $\gamma_1(t)$ and $\gamma_2(t)$ be the asymptotic geodesics in the boundary of E. Let $x < y$ be two consecutive times of intersection of $\gamma_1(t)$ with ∂N (here y may be $+\infty$). Then there exists a constant $A>0$ depending on the metric in M such that the area of E between $\gamma_1 : [x, y] \longrightarrow M$ and $\gamma_2 : [x, y] \longrightarrow M$ is bounded above by*

$$Ad(\gamma_1(x), \gamma_2(x)).$$

We shall subdivide the proof of Proposition 3.1 in several lemmas. We start by noting that

Claim: *Under our assumptions, if $\gamma_i(t_i) \in \partial N$ for some $t_i > 0$ and some $i = 1, 2$, then both $\gamma_1(t)$ and $\gamma_2(t)$ cross N at the same connected component of ∂N for some positive $t_1' > t_1$ and $t_2' > t_2$ respectively.*

Otherwise it is not hard to see that it would exist $t > 0$ such that $\gamma_1(t)$ is at distance at least $\frac{\omega}{2}$ from every point of γ_2 which contradicts the choice of α_1.

Given two differentiable curves c_1, c_2, let us denote by $\angle_p(c_1, c_2)$ the angle of the intersection $c_1 \cap c_2$ at the point p. Assume without loss of generality that $\gamma_1(t)$ crosses ∂N at $t = 0$. This claim allows us to consider three cases of crossings:

1. $\gamma_i(t)$ remains in N for some $i = 1, 2$ and every positive time.
2. The angle of crossing satisfies $0 < \angle_{\gamma_1(0)}(\gamma_1, \partial N) \leq \epsilon_0$.
3. The angle of crossing satisfies $\epsilon_0 < \angle_{\gamma_1(0)}(\gamma_1, \partial N) \leq \frac{\pi}{2}$.

Although the finiteness of the area in case 1 would follow from the results of section 1 and some more calculus, we are going to treat cases 1 and 3 together taking advantage of the symmetries of our example. First, notice that in case 1 there are again two possibilities:

1. The geodesic γ_i coincides with the zero curvature geodesic γ_0 for some $i = 1, 2$.
2. Both γ_1, γ_2 differ from γ_0.

The proposition in the hypotheses of item 1 is already proven by the results of section 1. The hypotheses of item 2 imply that the geodesics in the boundary of E are both asymptotic to γ_0. So by lemma 3.1 there exists an isometry $T : N \longrightarrow N$ preserving ∂N taking γ_1 into γ_2. In general, every geodesic segment in N gives rise to many isometric copies of it, via the symmetries of N. Remember that we are in $\bar{M}$, so from now on we shall fix the lifting of N - which we still denote by N for simplicity - containing the geodesic $\gamma_0 \subset \bar{M}$ and notice that the rotations of lemma 3.1 lift to isometries of $N \subset \bar{M}$ which act as translations.

Lemma 3.3. *Let $\gamma : [0, a] \longrightarrow N$ and $\beta : [0, a] \longrightarrow N$ be two geodesic segments (a may be $+\infty$) satisfying the following conditions:*
1. *There exists a rotation $T : N \longrightarrow N$ such that $T(\gamma) = \beta$.*
2. *Let L, L' be the segments in ∂N with endpoints $\gamma(0)$, $\beta(0)$ and $\gamma(a)$, $\beta(a)$ respectively. The region $S \subset N$ bounded by L, L', and the geodesics $\gamma([0, a])$, $\beta([0, a])$ is either diffeomorphic to a rectangle or diffeomorphic to $[0, 1] \times [0, +\infty)$.*

Then the area of S is bounded above by $H length(L)\omega$ where H is some constant depending on the metric.

Proof. First, notice that the hypotheses on γ and β imply that either $a = +\infty$ and they are asymptotic to γ_0, or they both intersect ∂N at its two boundary components. For, if the number $a < +\infty$ it is not hard to see that if γ, β intersect ∂N in only one boundary component then S is not diffeomorphic to a rectangle.

Now, let $Q \subset N$ be the rectangle with sides L, the symmetric image of L by the reflection of N fixing γ and the segments I, $T(I)$ of stable horospheres of γ_0 in N containing the endpoints of L. Q is a fundamental domain for the rotation which sends γ to β. Let $Q_0 = Q$ and let Q_i for $i = 1, 2, .., m$ be the iterates of Q_0 by the rotation, where m is the smallest integer with the property that $\cup_{i=0}^{m-1} Q_i$ covers S (of course, m may be $+\infty$). Let $B_i = S \cap Q_i$. Then we can fill up either Q (if

$a < +\infty$) or a half of Q (if $a = +\infty$) with isometric images of the B_i's by the iterates of T in a way that these images intersect two by two only at their boundaries. This implies that the area of S is less or equal than the area of Q which is, up to a factor close to 1, $length(L)\omega$. □

Corollary 3.1. *Let S be as in the last lemma. Then there exists a constant A_1 depending on the metric such that*

$$Area(S) \leq A_1 d(\gamma(0), \beta(0)).$$

Proof. This is just because the angle of intersection between γ and ∂N at the point $\gamma(0)$ is at least ϵ_0 by lemma 3.2. So there exists a constant P depending on the metric such that the length of L is bounded above by P times the length of the geodesic segment joining $\gamma(0)$ and $\beta(0)$. Or in other words, $length(L) \leq HPd(\gamma(0), \beta(0))$ which proves the corollary letting $A_1 = PH\omega$. □

This lemma completes the proof of Proposition 3.1 in case 1. Next, we shall show that case 3 also reduces to this lemma. The angle $\angle_{\gamma_1(0)}(\gamma_1, N_i)$ is greater than ϵ_0 and therefore, by the same reasoning of lemma 3.2 we deduce that $\gamma_1(0)$ and $\gamma_1(t_0)$ - the first positive time at which γ_1 leaves N - belong to different components of ∂N. Notice that in this case the end E actually intersects γ_0 and therefore it has points of zero curvature.

Suppose that $\gamma_1(0) \in N_1$ and $\gamma_1(t_0) \in N_2$. By lemma 2.1 there is a family of geodesic segments in N which are all isometric to $[\gamma_1(0), \gamma_1(t_0)]$ and come from the action of the rotations of the annulus over this geodesic segment. Let us rotate γ_1 in order to get a geodesic $\beta_1 : [0, t_0] \longrightarrow N$ isometric to $\gamma_1([0, t_0])$ and whose initial value is $\beta_1(0) = \gamma_2(s_0)$, where $\gamma_2([s_0, s_1])$ is the connected component of γ_2 in the boundary of $E \cap N$. Note that $\gamma_2(s_1) \in N_2$.

Lemma 3.4. *Suppose the hypotheses of case 3 hold. Then the strip S_0 in N bounded by $\gamma_1 : [0, t_0] \longrightarrow N$ and $\beta_1 : [0, t_0] \longrightarrow N$ contains $\gamma_2 : [s_0, s_1] \longrightarrow N$.*

Proof. Clearly, if β_1 was an asymptote of γ_1 then there would be nothing to prove. So let us suppose that β_1 is not asymptotic to γ_1. Recall that these geodesics are in the universal covering $\tilde{\mathrm{M}}$. The end E is a thin strip bounded by $\gamma_1(t)$ and $\gamma_2(t)$ whose width goes to zero if $t \to +\infty$. In fact $\beta_1(t)$ and $\bar{\gamma}_2(t) = \gamma_2(t+s_0)$ are two geodesic rays starting from the same point and making a small angle at this point. Consider the region V in $\tilde{\mathrm{M}}$ bounded by $\gamma_1(t)$, $\beta_1(t)$ for $t \geq 0$ and the small segment of N_1 with endpoints $\gamma_1(0), \beta_1(0)$. Since β_1 is not asymptotic to γ_1 by assumption we have that the region V is a part of a non-compact cone with infinite volume. So let us suppose that $\gamma_2(t)$ is not contained in the strip S_0. Then it must intersect $\beta_1(t)$ eventually in the future because it is asymptotic to γ_1 and β_1 diverges from γ_1. This means that γ_2 and β_1 have two different points of intersection in $\tilde{\mathrm{M}}$ which is not allowed by the geometry of non-positively curved manifolds. This concludes the proof of the lemma. □

Therefore, we can estimate the area of the intersection $E \cap N$ in case 3 by lemma 3.3 and corollary 3.1 taking $T = t_0$, $\gamma = \gamma_1$, $\beta = \beta_1$ and $S = S_0$, since the strip S_0 contains $E \cap N$ in this case. This finishes the proof of Proposition 3.1 in case 3.

To estimate the area in case 2 recall that from lemma 3.2 we have that $\gamma_1(t)$ enters and leaves N for the first time through the same connected component of ∂N. Let $\gamma_1 : [0, t_0] \longrightarrow N$ and $\gamma_2 : [s_0, s_1] \longrightarrow N$ be, as above, the connected components of γ_1 and γ_2 in $E \cap N$.

Lemma 3.5. *Assume that we are in the hypothesis of case 2. There exists a constant A_2 such that the area of the connected piece of $E \cap \partial N$ whose boundary contains $[\gamma_1(0), \gamma_1(t_0)]$ is bounded above by*

$$d(\gamma_1(0), \gamma_2(0))A_2.$$

Proof. Let I be the segment in the stable horosphere of $\gamma_1(0)$ bounded by $\gamma_1(0), \gamma_2(0)$. Notice that, since this horosphere is normal to γ_1 and the angle $\angle_{\gamma_1(0)}(\gamma_1, N_1)$ is smaller than ϵ_0 then the segment I is not in N and I is at distance at most ω from γ_0. So from lemma 3.1 we have that every point of I is contained in an asymptote of γ_0 which remains at distance less than ω for all $t \geq 0$.

Consider the strip S formed by these stable asymptotes of γ_0, bounded by a segment of stable horosphere of γ_0. We shall prove that the area of E is less or equal than the area of S. Parametrize I in [1,2] by arclength, let us take $I(1) = \gamma_1(0)$ and $I(2) = \gamma_2(0)$. Denote by $\gamma_r, r \in [1,2]$ the asymptotes of γ_1 contained in E and let $\beta_r, r \in [1,2]$ be the asymptote of γ_0 containing the point $I(r)$. Denote by $J^s(\alpha(t))$ the stable Jacobi field (it is unique up to orientation) defined along a geodesic $\alpha(t)$ with initial condition $\mid J^s(\alpha(0)) \mid = 1$. Then, the area of the strip S is

$$\int_1^2 \int_0^{+\infty} \mid J^s(\beta_r(t)) \mid dt\, dr.$$

Now, recalling that if $\gamma_1(t_0)$ is the first positive exit of γ_1 from N we have that

$$d(\beta_r(t), \gamma_0) < d(\gamma_r(t), \gamma_0)$$

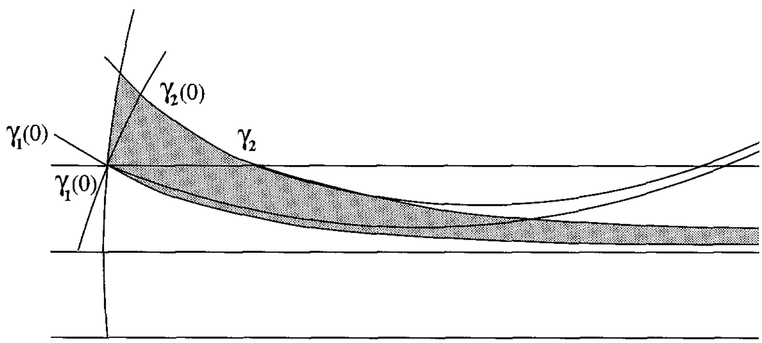

Figure 1

for every $t \in [0, t_0]$ and $r \in [1,2]$. This is because the angle of intersection of γ_r with N_1 is smaller than ϵ_0, the angle of intersection of the asymptotes of γ_0, and therefore $\beta_r(t)$ remains always closer to γ_0 than $\gamma_r(t)$ (see the proof of lemma 3.2). From the construction, this implies that the curvature at $\beta_r(t)$ is bigger than the curvature at $\gamma_r(t)$ for such t's and therefore, Rauch comparison theorem tells us that

$$\mid J^s(\gamma_r(t)) \mid \leq \mid J^s(\beta_r(t)) \mid$$

for every $t \in [0, t_0], r \in [1, 2]$ so we get that

$$\int_1^2 \int_0^{t_0} \mid J^s(\gamma_r(t)) \mid dtdr \leq \int_1^2 \int_0^{+\infty} \mid J^s(\beta_r(t)) \mid dt\, dr$$
$$\leq D\, length(I) A_1,$$

where D is some constant depending on the metric and A_1 is the constant obtained corollary 3.1. It is not hard to see that we can assume, without loss of generality, that $\gamma_r(t_0)$ does not belong to N for every $r > 1$ (otherwise, take γ_2 instead of γ_1 to argue). In this way, the region $\bar{E}$ bounded by $\gamma_1([0, t_0])$, $\gamma_2([0, t_0])$ and the horospheric segments joining $\gamma_1(0)$, $\gamma_2(0)$ to $\gamma_1(t_0)$, $\gamma_2(t_0)$ respectively, contains E. The above two inequalities imply

$$\begin{aligned} Area(E) &\leq Area(\bar{E}) \\ &\leq \int_1^2 \int_0^{t_0} \mid J^s(\gamma_r(t)) \mid dt\, dr \\ &\leq DA_1\, length(I). \end{aligned}$$

Thus, the area of E which is bounded above by the left hand side of the above inequality satisfies the statement of the lemma. □

4. Estimates outside the critical region and the proof of the main Theorem

Let $\gamma(t)$ be a geodesic of M and let $\bar{J}(t)$ be a perpendicular Jacobi field defined along γ. The norm $J(t)$ of $\bar{J}(t)$ satisfies the Jacobi equation

$$J''(t) + K(t)J(t) = 0,$$

where $K(t)$ is the Gaussian curvature at $\gamma(t)$. The function $u(t) = \frac{J'(t)}{J(t)}$ satisfies the well-known Ricatti equation

$$u'(t) + u^2(t) + K(t) = 0.$$

Lemma 4.1. *Assume that $K(t) \leq -a < 0$ for every $t \in [0, T]$. Then every solution $u(t)$ of the Ricatti equation with $u(t) \leq 0$ $\forall t \in [0, T]$ satisfies*

$$u(t) \leq \max\{-\frac{a}{2}(T - t), -\sqrt{\frac{a}{2}}\}$$

Proof. Let $t \in [0,T]$. If $u(t) > -\sqrt{\frac{a}{2}}$ we have that

$$u'(t) = -K(t) - u^2(t) > a - \frac{a}{2} = \frac{a}{2}.$$

Thus $u(s)$ is increasing when $-\sqrt{\frac{a}{2}} < u(s) \leq 0$. In particular, if $u(t) > -\sqrt{\frac{a}{2}}$ then $u(s) > -\sqrt{\frac{a}{2}}$ and $u'(s) > \frac{a}{2}$ for every $s \in [t,T]$. We have that

$$\begin{aligned} u(t) &= u(T) + \int_T^t u'(s)ds \\ &\leq \int_t^T -u'(s)ds \\ &\leq \int_t^T -\frac{a}{2}ds \\ &= -\frac{a}{2}(T-t) \end{aligned}$$

from which we conclude the statement. $\square$

Corollary 4.1. *Let $K(t) \leq -a < 0$ for every $t \in [0,T]$. If $u(t) \leq 0$ $\forall t \in [0,T]$ where $u(t)$ is a solution of the Ricatti equation, then for all $0 \leq \delta \leq \sqrt{\frac{2}{a}}$ we have that*

1. *$u(t) \leq -\frac{a}{2}(T-t)$ if $T-\delta \leq t \leq T$.*
2. *$u(t) \leq -\frac{a}{2}\delta$ if $0 \leq t \leq T-\delta$.*

Proof. From lemma 4.1 we have that

$$u(t) \leq -\frac{a}{2}(T-t)$$

if $T - t \leq -\sqrt{\frac{2}{a}}$ and

$$u(t) \leq -\sqrt{\frac{a}{2}}$$

if $T-t \geq -\sqrt{\frac{2}{a}}$. $\square$

Lemma 4.2. *Let $K(t) \leq -a < 0$ for every $t \in [0,T]$. Suppose that $J(t)$ is a perpendicular Jacobi field defined along $\gamma(t)$ with $J'(t) \leq 0$ for every $t \in [0,T]$. Then for all $0 < \delta < \sqrt{\frac{2}{a}}$ we have that*

1. *$J(T) \leq J(0)\exp(-\frac{a}{4}T^2)$ if $0 < T < \delta$.*
2. *$J(T) \leq J(0)\exp(-\frac{a}{2}\delta(T-\frac{\delta}{2}))$ if $\delta < T$.*

Proof. We have that $\frac{d}{dt}\log(J(t)) = \frac{J'(t)}{J(t)} = u(t) \leq 0$ for all $t \in [0,T]$. Since $u(t)$ satisfies the Ricatti equation, corollary 4.1 proceeds and then for $0 < T < \delta$ we get

$$\begin{aligned}
\log(\frac{J(T)}{J(0)}) &= \int_0^T u(s)ds \\
&\leq \int_0^T -\frac{a}{2}(T-s)ds \\
&\leq \int_0^T -\frac{ax}{2}dx \\
&= -\frac{a}{4}T^2
\end{aligned}$$

This implies that

$$J(T) \leq J(0)\exp(-\frac{a}{4}T^2)$$

if $0 < T < \delta$. Moreover, if $T > \delta$ then

$$\begin{aligned}
\log(\frac{J(T)}{J(0)}) &= \int_0^T u(s)ds \\
&= \int_{T-\delta}^T u(s)ds + \int_0^{T-\delta} u(s)ds \\
&\leq \int_{T-\delta}^T -\frac{a}{2}(T-s)ds + \int_0^{T-\delta} -\frac{a\delta}{2}ds \\
&\leq \int_0^\delta -\frac{ax}{2}dx - \frac{a\delta}{2}(T-\delta) \\
&\leq -\frac{a}{4}\delta^2 + \frac{a}{2}\delta^2 - \frac{a}{2}\delta T \\
&= -\frac{a\delta}{2}(T-\frac{\delta}{2})
\end{aligned}$$

which implies that

$$J(T) \leq J(0)\exp(-\frac{a\delta}{2}(T-\frac{\delta}{2}))$$

as we wanted to show. □

Corollary 4.2. *Let $K(t)$ and $J(t)$ be as in lemma 4.2. Then for every $0 < \delta < \sqrt{\frac{2}{a}}$ there exists $0 < \mu(\delta) < 1$, $Q(\delta) > 0$ such that if $T > \delta$ then*

$$J(T) < \mu(\delta)J(0)$$

and

$$\int_0^T J(s)ds \leq Q(\delta)J(0).$$

Proof. Let $\mu(\delta) = \exp(-\frac{a}{4}\delta^2)$. The first inequality in the statement follows from lemma 4.2. Moreover,

$$\begin{aligned}\int_0^T J(s)ds &\leq \int_0^T J(0)\exp\left(\frac{a}{4}\delta^2 - \frac{a}{2}\delta s\right) ds \\ &\leq J(0)\exp\left(\frac{a}{4}\delta^2\right)\frac{2}{a\delta}\left[1-\exp\left(-\frac{a}{2}\delta T\right)\right] \\ &\leq Q(\delta)J(0)\end{aligned}$$

where $Q(\delta) = \frac{2}{a\delta}\exp(\frac{a}{4}\delta^2)$. □

We shall preserve the notations of the previous section. So let E be a thin end having (horospheric) width less than the constant α_1 defined in section 3. Let $\gamma_1(t), \gamma_2(t)$ be the pair of asymptotic geodesics in the boundary of E parameterized by arclength in a way that the segments of stable horospheres of $\gamma_1(t)$ in E intersect γ_2 at $\gamma_2(t)$. We are going to estimate the area of the pieces of E in the complement of the region of M containing the closed geodesic of zero curvature. Again we identify E with its projection in M by the covering map. As before, let $N = N_\omega$ be the tubular neighborhood of the closed geodesic γ_0 of radius the constant ω defined at the beginning of section 3. Let us make some choices:

1. Recalling that $\alpha_1 < \frac{\omega}{4}$ let $a > 0$ be such that if p, q are any two points in the same stable horosphere, $p \in M - N$ and their horospheric distance $d^{ss}(p,q) < \alpha_1$, then the curvature at q is $K(q) < -a$.
2. Fix $\delta > 0$ such that if $\gamma(t)$ is a geodesic of M with $\gamma(0) \in \partial(N)$ and $\gamma'(0)$ points outwards N, then $\gamma(t)$ does not belong to N for every $0 < t < 2\delta$.
3. Fix $\mu = \mu(\delta) < 1$ and $Q = Q(\delta)$ from corollary 4.2.

Lemma 4.3. *Assume that $\gamma_1(t)$ does not belong to N for every $0 < t < T$, with $T > \delta$. Then*

$$d^{ss}(\gamma_1(T), \gamma_2(T)) \leq \mu d^{ss}(\gamma_1(0), \gamma_2(0))$$

and

$$A(T) \leq Qd^{ss}(\gamma_1(0), \gamma_2(0))$$

where $A(T)$ is the area of the region bounded by $\gamma_1(t), \gamma_2(t)$ for $0 \leq t \leq T$ and the stable horospheric segments $[\gamma_1(0), \gamma_2(0)]$ and $[\gamma_1(T), \gamma_2(T)]$.

Proof. The argument is similar to the one used in lemma 3.5. The Buseman flow of γ_1 provides a differentiable parameterization $f : [0, l] \times [0, T] \longrightarrow M$ of the subset of E considered in the statement of the lemma. This parameterization is such that the curve $t \to f_s(t) = f(s, t)$ is an asymptote of $\gamma_1(t)$ $\forall s \in [0, 1]$ and $s \to f(s, t)$ is a parameterization of the horospheric segment joining $\gamma_1(t)$ and $\gamma_2(t)$. Moreover, the curve $s \to f(s, 0)$ is the arc length parameterization of the horospheric segment joining $\gamma_1(0)$ and $\gamma_2(0)$. The function $\bar{J}_s(t) = \frac{df}{dt}(s, t)$ is a Jacobi field perpendicular to the geodesic $f_s(t)$ satisfying $\bar{J}_s(t) = J_s(t)e(t)$, where $J_s(t) =\mid \bar{J}_s(t) \mid$ and $e(t)$ is a unitary vector field tangent to the horosphere containing $\gamma_1(t)$. Since the curvature of M is non-positive the functions $J_s(t)$ are non-increasing for every s, t. Thus, by corollary 4.2 we get

$$J_s(T) \leq \mu J_s(0)$$

$$\int_0^T J_s(t)dt \leq QJ_s(0).$$

Now,

$$\begin{aligned} d^{ss}(\gamma_1(T), \gamma_2(T)) &= \int_0^l J_s(T)ds \\ &\leq \mu \int_0^l J_s(0)ds \\ &= \mu d^{ss}(\gamma_1(0), \gamma_2(0)). \end{aligned}$$

Also,

$$\begin{aligned} A(T) &= \int_0^l \int_0^T \mid Jacobian(Df(s, t)) \mid dt\, ds \\ &= \int_0^l \int_0^T J_s(t)dt\, ds \\ &\leq \int_0^l QJ_s(0)ds \\ &= Qd^{ss}(\gamma_1(0), \gamma_2(0)), \end{aligned}$$

which finishes the proof of the lemma. □

Given an interval $I = [a, b]$ let $A(I)$ be the area of the region bounded by $\gamma_1(t)$, $\gamma_2(t)$ for $t \in [a, b]$ and the horospheric segments joining $\gamma_1(a)$ to $\gamma_2(a)$ and $\gamma_1(b)$ to $\gamma_2(b)$. The following result completes the proof of the main theorem:

Lemma 4.4. *There exists $A > 0$ such that $A([0, +\infty)) \leq A$.*

Proof. Let $t_0 = 0$ and let $0 \leq t_i < s_i < t_{i+1}, i \geq 1$ be the sequence of times where $\gamma_1(t_i) \in \partial N$, for every i the segment $\gamma_1 : [t_i, s_i] \longrightarrow M$ is a subset of the closure of N and $\gamma_1 : [s_i, t_{i+1}] \longrightarrow M$ is a subset of the complement of N. Let $d_n = d^{ss}(\gamma_1(t_n), \gamma_2(t_n))$ and $e_n = d^{ss}(\gamma_1(s_n), \gamma_2(s_n))$. From section 3 we have that

$$A([t_n, s_n]) \leq B d_n$$

for some $B > 0$. By lemma 4.3,

$$A([s_n, t_{n+1}]) \leq Q e_n$$

$$d_n \leq \mu e_n.$$

Since distances between asymptotic geodesics are decreasing we have that $e_n < d_n$, and therefore $e_{n+1} \leq \mu e_n$ and $d_{n+1} \leq \mu d_n$. And since $e_0 \leq d_0 \leq \alpha_1$ we deduce

$$\begin{aligned} A([0, +\infty)) &= \sum_n A([t_n, s_n]) + A([s_n, t_{n+1}]) \\ &\leq \sum_n (B d_n + Q e_n) \\ &\leq (B + Q) \sum_n 2\mu^n \alpha_1 \\ &\leq \frac{2(B + Q)}{1 - \mu} \alpha_1, \end{aligned}$$

which ends the proof of the lemma. □

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